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On Nonlinear Integro-differential Operators in Generalized Orlicz–Sobolev Spaces

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A nonlinear integral operator T of the form

$$(Tf)(s) = \int_G K(t, f(\sigma(s, t)))\, d\mu(t),$$

for $s \in G$, is defined and investigated in the measure space (G, Σ, μ), where f and K are vector-valued functions with values in normed linear spaces E and F, respectively. The results are applied to the case of integro-differential operators in generalized Orlicz–Sobolev spaces. There are studied problems of existence, embeddings, and approximation by means of T. © 2000 Academic Press

Key Words: approximation by strongly singular integrals; Orlicz–Sobolev space; modular space; nonlinear integro-differential operator; generalized Lipschitz condition.

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1. NOTATIONS AND DEFINITIONS

Let E, F be two normed linear spaces with norms $\| \|_E$, $\| \|_F$, respectively, and let G be a nonempty abstract set with a commutative operation "$+$" such that if $s, t \in G$ then $s+t \in G$. We assume that (G, Σ, μ) is a measure space with a σ-finite, complete measure μ. Let μ_π be the completion of the product measure $\mu \times \mu$ in $G \times G$ defined on the completion of the product σ-algebra, denoted by Σ_π.

We shall say that the product measure μ_π is *absolutely continuous* with respect to μ if for every set $A \in \Sigma$ such that $\mu(A)=0$ there holds $\mu_\pi(A_{-1}) = 0$ where $A_{-1} = \{(x, y) \in G \times G : x+y \in A\}$. We shall denote by $L^0(G, E)$ (resp. $L^0(G, F)$), the space of all strongly Σ-measurable vector-valued functions f: $G \to E$ (resp. g: $G \to F$) with equality μ-almost everywhere. We shall write also $L^0(G) = L^0(G, \mathbb{R})$. $L^1(G, E)$ (resp. $L^1(G, F)$) will mean the space of Bochner integrable functions $f \in L^0(G, E)$ (resp. $L^0(G, F)$) with respect to the measure μ. Let $\mathscr{A} \in \mathscr{L}(E, F)$; i.e., $\mathscr{A}$ is a linear continuous operator from E to F.

We shall say that K: $G \times E \to F$ is a *kernel function* with respect to $\mathscr{A}$ if $\mathscr{A}u = 0$ implies $K(t, u) = 0$ for all $t \in G$, $u \in E$ and $K(\cdot, u) \in L^1(G, F)$ for all $u \in E$. Let L: $G \to \mathbb{R}_0^+ = [0, +\infty[$, $L \in L^1(G) = L^1(G, \mathbb{R})$; in the following we assume $D = \int_G L(t)\, d\mu(t) > 0$ and we put $p(t) = L(t)\, D^{-1}$. Let ψ: $G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy the following conditions: $\psi(\cdot, x)$ is Σ-measurable for all $x \geqslant 0$; $\psi(t, \cdot)$ is continuous and nondecreasing, $\psi(t, 0) = 0$, $\psi(t, x) > 0$, for $x > 0$, $\psi(t, x) \to +\infty$ as $x \to +\infty$, for all $t \in G$. If additionally $\psi(t, \cdot)$ is concave for all $t \in G$ we say that ψ is *concave*. A kernel function K will be called $(L, \psi)_0$-*Lipschitz* if there holds the inequality $\|K(t, u)\|_F \leqslant L(t)\, \psi(t, \|u\|_E)$, for all $t \in G$, $u \in E$. K will be called (L, ψ)-*Lipschitz* if there holds the inequality

$$\|K(t, u) - K(t, v)\|_F \leqslant L(t)\, \psi(t, \|u - v\|_E),$$

for all $t \in G$, $u, v \in E$ (for these notions see [1, 2, 10, 11]).

We define an operator T by the formula

$$(Tf)(s) = \int_G K(t, f(s+t))\, d\mu(t)$$

for all functions $f \in L^0(G, E)$ such that the above Bochner integral exists for μ—a.e. $s \in G$, and $Tf \in L^0(G, F)$. The set of all such functions f will be denoted by Dom T and called the *domain of the operator* T.

Let η, ρ be two modulars on $L^0(G, E)$, $L^0(G, F)$, respectively, and let $\tilde{\rho}$ be a modular on $L^0(G)$. We suppose that $\rho(f) = \tilde{\rho}(\|f\|_F)$, for $f \in L^0(G, F)$. We recall here that a *modular* on a real vector space X is a functional

$\gamma: X \to [0, +\infty]$ such that $\gamma(x) = 0$ if and only if $x = 0$; $\gamma(x) = \gamma(-x)$, for every $x \in X$; $\gamma(ax + by) \leqslant \gamma(x) + \gamma(y)$, for every $x, y \in X$, and $a, b \in \mathbb{R}_0^+$, $a + b = 1$. We note that the last property implies that for any modular γ we have $\gamma(ax) \leqslant \gamma(bx)$, for every $x \in X$, and $a, b \in \mathbb{R}_+$, with $a \leqslant b$.

The modular space generated in $L^0(G, E)$ by η will be denoted by $L^0_\eta(G, E)$, being

$$L^0_\eta(G, E) = \{ f \in L^0(G, E) : \lim_{\lambda \to 0} \eta(\lambda f) = 0 \};$$

analogously by $L^0_\rho(G, F)$ we will denote the modular space generated in $L^0(G, F)$ by ρ, (see [9]). The modular $\tilde{\rho}$ is called *$\mathscr{J}$-convex* if for every two measurable functions $p: G \to \mathbb{R}_0^+$, with $\int_G p(t)\, d\mu(t) = 1$ and $\xi: G \times G \to \mathbb{R}$, the inequality

$$\tilde{\rho}\left(\int_G p(t)\, |\xi(t, \cdot)|\, d\mu(t) \right) \leqslant \int_G p(t)\, \tilde{\rho}(|\xi(t, \cdot)|)\, d\mu(t)$$

holds (see [2]).

The modular $\tilde{\rho}$ is called *quasimonotone* if there is a constant $M \geqslant 1$ such that if $f_1, f_2 \in L^0(G)$ with $|f_1| \leqslant |f_2|$, then $\tilde{\rho}(f_1) \leqslant M \tilde{\rho}(M f_2)$.

The modular η is called *τ-subbounded* if there exist constants $c_1, c_2 \geqslant 1$, and $h_0 \geqslant 0$ such that

$$\eta(f(t + \cdot)) \leqslant c_1 \eta(c_2 f) + h_0$$

for all $f \in L^0(G, F)$.

We call $\{\tilde{\rho}, \psi, \eta\}$ a *properly directed triple* if there is a set $G_0 \subset G$, $G_0 \in \Sigma$, $\mu(G \backslash G_0) = 0$ such that for every $\lambda \in\,]0, 1[$ there exists a $C_\lambda \in\,]0, 1[$ satisfying the inequality

$$\tilde{\rho}[C_\lambda \psi(t, \|\xi(\cdot)\|_E)] \leqslant \eta[\lambda \xi(\cdot)],$$

for all $t \in G_0$ and $\xi \in L^0(G, E)$. This implies the inequality

$$\tilde{\rho}[C_\lambda \psi(t, \|\xi_t(\cdot)\|_E)] \leqslant \eta[\lambda \xi_t(\cdot)]$$

for all $t \in G_0$ and any family $(\xi_t(\cdot))_{t \in G}$ of functions $\xi_t \in L^0(G, E)$ (see [1, 2]).

2. AN ESTIMATE FOR THE ERROR OF APPROXIMATION

We are going to estimate $\rho(a(Tf - \mathscr{A} \circ f))$, for $f \in \operatorname{Dom} T$, $a > 0$, by means of η. We shall need the notion of *η-modulus continuity* ω_η of a function

$f \in L^0(G, E)$. Let $\mathscr{U}$ be a nonempty family of sets $U \in \Sigma$, $U \neq \emptyset$. Then ω_η: $L^0(G, E) \times \mathscr{U} \to \overline{\mathbb{R}}_0^+ = [0, +\infty]$ is defined by the formula

$$\omega_\eta(f, U) = \sup_{t \in U} \eta(f(t + \cdot) - f(\cdot))$$

(see [3, 8]). There holds the following:

THEOREM 1. *Let η, ρ be modulars on $L^0(G, E)$, $L^0(G, F)$, respectively, and let $\tilde{\rho}$ be quasimonotone with a constant $M \geqslant 1$ and $\mathscr{J}$-convex modular on $L^0(G)$. Moreover, let ρ be of the form $\rho(f) = \tilde{\rho}(\|f\|_F)$ and let τ-subbounded with constants c_1, $c_2 \geqslant 1$, $h_0 \geqslant 0$. Let $\{\tilde{\rho}, \psi, \eta\}$ be a properly directed triple and let K be an (L, ψ)-Lipschitz kernel function. Then for every $f \in \mathrm{Dom}\, T$, $U \in \mathscr{U}$, $\lambda \in]0, 1[$, and $a \in]0, C_\lambda(2DM)^{-1}[$, there holds the inequality*

$$\rho(a(Tf - \mathscr{A} \circ f)) \leqslant M\omega_\eta(\lambda f, U) + M\{2c_1 \eta(2\lambda c_2 f(\cdot)) + h_0\} \times \int_{G \setminus U} p(t)\, d\mu(t) + M\tilde{\rho}(2aMr_0 \|\mathscr{A} \circ f\|_F), \tag{1}$$

where $D = \int_G L(t)\, d\mu(t)$, $p(t) = L(t)\, D^{-1}$, and

$$r_0 = \sup_{u \in E,\, \mathscr{A}u \neq 0} \frac{1}{\|\mathscr{A}u\|_F} \left\| \int_G K(t, u)\, d\mu(t) - \mathscr{A}u \right\|_F.$$

Proof. Obviously we can assume that $f \in L^0_\eta(G, E)$, otherwise the right-hand side of (1) would be infinite. By the properties of the modular ρ we have, for any $a > 0$,

$$\rho[a(Tf - \mathscr{A} \circ f)] \leqslant \rho\left[2a \int_G \{K(t, f(t + \cdot)) - K(t, f(\cdot))\}\, d\mu(t)\right] + \rho\left[2a \int_G K(t, f(\cdot))\, d\mu(t) - \mathscr{A} \circ f(\cdot)\right] = J_1 + J_2.$$

By the assumption that K is (L, ψ)-Lipschitz, we get

$$\left\| \int_G \{K(t, f(t + \cdot)) - K(t, f(\cdot))\}\, d\mu(t) \right\|_F \leqslant \int_G L(t)\, \psi(t, \|f(t + \cdot) - f(\cdot)\|_E)\, d\mu(t)$$

and so by applying quasimonotonicity and $\mathscr{J}$-convexity of $\tilde{\rho}$, we have

$$J_1 \leqslant M\tilde{\rho}\left[2aM\int_G L(t)\,\psi(t, \|f(t+\cdot)-f(\cdot)\|_E)\,d\mu(t)\right]$$

$$\leqslant M\int_G p(t)\,\tilde{\rho}[2aDM\psi(t, \|f(t+\cdot)-f(\cdot)\|_E)]\,d\mu(t).$$

By assumption that $\{\tilde{\rho}, \psi, \eta\}$ is properly directed, for every $\lambda \in\,]0, 1[$ and for $a \in\,]0, C_\lambda(2DM)^{-1}[$ we deduce, for $U \in \mathscr{U}$,

$$J_1 \leqslant M\int_G p(t)\,\eta[\lambda(f(t+\cdot)-f(\cdot))]\,d\mu(t)$$

$$\leqslant M\omega(\lambda f, U) + M\int_{G\setminus U} p(t)\,\eta[\lambda(f(t+\cdot)-f(\cdot))]\,d\mu(t) = J_1^1 + J_1^2.$$

Now by the properties of the modular η and from τ-subboundedness of η, we get

$$J_1^2 \leqslant M\int_{G\setminus U} p(t)\,\eta[2\lambda(f(t+\cdot))]\,d\mu(t) + M\int_{G\setminus U} p(t)\,\eta[2\lambda f(\cdot)]\,d\mu(t)$$

$$\leqslant Mc_1\int_{G\setminus U} p(t)\,\eta[2\lambda c_2 f(\cdot)]\,d\mu(t) + Mh_0\int_{G\setminus U} p(t)\,d\mu(t)$$

$$+ M\eta(2\lambda f(\cdot))\int_{G\setminus U} p(t)\,d\mu(t)$$

$$\leqslant [2Mc_1\eta(2\lambda c_2 f) + Mh_0]\int_{G\setminus U} p(t)\,d\mu(t).$$

Thus we obtain the estimation for J_1:

$$J_1 \leqslant M[2c_1\eta(2\lambda c_2 f) + h_0]\int_{G\setminus U} p(t)\,d\mu(t) + M\omega_\eta(\lambda f, U).$$

Finally putting $G_1 = \{s \in G : \mathscr{A} \circ f(s) \neq 0\}$, we have

$$J_2 = \tilde{\rho}\left[2a\left\|\int_G K(t, f(\cdot))\,d\mu(t) - (\mathscr{A}\circ f)(\cdot)\right\|_F \chi_{G_1}(\cdot)\right]$$

$$\leqslant M\tilde{\rho}[2aMr_0\,\|\mathscr{A}\circ f\|_F]$$

and so the inequality (1) follows.

3. A MEASURABILITY RESULT

We now explain in detail the notions introduced in Section 2. Let $G \subset \mathbb{R}$ be a compact interval $[a, b]$ or $G = \mathbb{R}$. In the first case we denote by $\mathscr{L}^0(G)$ the set of all the $(b-a)$-periodic functions $f \in L^0(\mathbb{R})$. In the second case $\mathscr{L}^0(G)$ will mean the set of all functions $f \in L^0(\mathbb{R})$ which are of bounded support. Let μ be the Lebesgue measure on the σ-algebra of all Lebesgue measurable subsets of G.

Let σ: $G \times G \to G$ be a (σ_π, Σ)-measurable function. We will say that μ_π is σ-absolutely continuous with respect to μ if for every set $A \in \Sigma$ such that $\mu(A) = 0$ there holds $\mu_\pi(\sigma^{-1}(A)) = 0$. There holds the following:

PROPOSITION 1. *Let E be an arbitrary measurable subset of $\mathbb{R}^n$, $F = \mathbb{R}$ and let σ: $G \times G \to G$ be $(\Sigma_\pi n \Sigma)$-measurable. Let μ_π be σ-absolutely continuous with respect to μ. Let K: $G \times E \to \mathbb{R}$ be a function which is measurable with respect to $t \in G$ for every $u \in E$ and continuous in $u \in E$ for every $t \in G$. Then the function $\mathscr{K}: G \times E \to \mathbb{R}$ defined by $\mathscr{K}(s, t) = K(t, f(\sigma(s, t)))$ for s, $t \in G$ is Σ_π-measurable in $G \times G$.*

Proof. We limit ourselves to the case $n = 1$. Let $f \in \mathscr{L}^0(G)$. Suppose $G = \mathbb{R}$, so that f has a bounded support. Let us denote by $[a, b]$ an interval such that supp $f \subset [a, b]$. By Fréchet's theorem there is a sequence (f_k) of continuous functions on $[a, b]$, convergent to f a.e. in $[a, b]$ (see, e.g., [5]). Denoting by A the set of $t \in G$ for which the sequence $(f_k(t))$ does not converge to $f(t)$, we have $\mu(A) = 0$. Hence $\mu_\pi(\sigma^{-1}(A)) = 0$. If $(s, t) \notin \sigma^{-1}(A)$, i.e., $\sigma(s, t) \notin A$, then $f_k(\sigma(s, t)) \to f(\sigma(s, t))$ as $k \to +\infty$, whence $f_k(\sigma(s, t)) \to f(\sigma(s, t))$ as $k \to \infty$ μ_π-a.e. in $G \times G$. Since $K(t, u)$ is a continuous function of $u \geqslant 0$ for every $t \in G$, we obtain

$$K(t, f_k(\sigma(s, t))) \to K(t, f(\sigma(s, t)))$$

μ_π-a.e. in $G \times G$. This reduces the proof to the case of continuous functions, since if $K(:, f_k(\sigma(\cdot, :)))$ are proved to be Σ_π-measurable, the same also holds for the function $K(:, f(\sigma(\cdot, :)))$ by virtue of the completeness of μ_π in $G \times G$. So let us now suppose f to be continuous in G. Let (σ_i) be a sequence of simple functions in $(G \times G, \Sigma_\pi, \mu_\pi)$, convergent to σ for all $(s, t) \in G \times G$; i.e., there are pairwise disjoint sets $A_1^{(i)}, ..., A_{\nu_i}^{(i)} \in \Sigma_\pi$ and constants $c_1^{(i)}, ..., c_{\nu_i}^{(i)} \in \mathbb{R}$ such that $\sigma_i(s, t) = \sum_{j=1}^{\nu_i} c_j^{(i)} \chi_{A_j^{(i)}}(s, t)$ converges to $\sigma(s, t)$ for all s, $t \in G$. By continuity of f and also of $K(t, \cdot)$, we obtain:

$$K(t, f(\sigma(s, t))) = \lim_{i \to +\infty} K(t, f(\sigma_i(s, t)))$$

$$= \lim_{i \to +\infty} \sum_{j=1}^{\nu_i} K(t, f(c_j^{(i)})) \chi_{A_j^{(i)}}(s, t).$$

But the function at the right-hand side of the above equality is Σ_π-measurable in $G \times G$. Thus $K(:, f(\sigma(\cdot, :)))$ is Σ_π-measurable.

EXAMPLE 1. If $\sigma(s, t) = s + t$, (eventually extended by periodicity to $\mathbb{R}^2$), then μ_π is σ-a.c. with respect to the Lebesgue measure on G.

4. AN EXISTENCE THEOREM FOR THE OPERATOR T IN ORLICZ–SOBOLEV SPACES

Let $AC^{(n-2)}(G)$ be the space of all functions $f: G \to \mathbb{R}$ of compact support in $\mathbb{R}$ (in case of $G = \mathbb{R}$), or $(b-a)$-periodic in $\mathbb{R}$ (in case $G = [a, b]$), possessing absolutely continuous derivatives up to the order $n-2$, inclusively. We denote by E the set of points of $\mathbb{R}^n$ of the form $D_n f(t) = (f(t), f'(t), ..., f^{(n-2)}(t), f^{(n-1)}(t))$, where $f^{(n-1)}$ exists a.e. in G and is integrable on G. By Proposition 1 and Example 1, the function

$$\mathscr{K}(s, t) = K(t, D_n f(s+t)) = K(t, f(s+t), f'(s+t), ..., f^{(n-1)}(s+t))$$

is Σ_π-measurable in $G \times G$. Hence the integral

$$\int_G |K(t, D_n f(s+t))| \, dt$$

exists for a.e. $s \in G$; however, it may be infinite. Now, let $\varphi: G \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a φ-function depending on a parameter, i.e., $\varphi(\cdot, u)$ is measurable for all $u \geq 0$, $\varphi(t, 0) = 0$, $\varphi(t, u) > 0$ for $u > 0$, $\varphi(t, \cdot)$ is continuous and nondecreasing, for all $t \in G$. If $\varphi(t, \cdot)$ is convex, for every $t \in G$ and $\varphi(t, u)\, u^{-1} \to 0$ as $u \to 0$ as $u \to 0$, $\varphi(t, u)\, u^{-1} \to +\infty$ as $u \to +\infty$, the φ-function is said to be an N-function. For any φ-function φ, the modular

$$\eta(f) = \sum_{k=0}^{n-1} \int_G \varphi(t, |f^{(k)}(t)|) \, dt \tag{2}$$

defines a modular space $AC_\eta^{(n-2)}(G)$ denoted usually by $W_{n-1}^\varphi(G)$ and called the generalized Orlicz–Sobolev space generated by φ. Obviously η may be extended as a modular defined on the set $L^0(G, \mathbb{R}^n)$ of all measurable vector-valued functions $\bar{f}: G \to \mathbb{R}^n$ by the formula

$$\bar{\eta}(\bar{f}) = \sum_{k=0}^{n-1} \int_G \varphi(t, |f_k(t)|) \, dt,$$

where $\bar{f}(t) = (f_0(t), ..., f_{n-1}(t))$, for $t \in G$.

We have then for $f \in W_{n-1}^\varphi(G)$ the relation $\eta(f) = \bar{\eta}(D_n f)$.

We shall need the notion of τ-boundedness of the function φ. The function φ is called *weakly τ-bounded* if there exist a constant $c \geqslant 1$ and a measurable function ξ: $G \times G \to \mathbb{R}_0^+$ such that $\sup_{s \in G} \int_G \xi(s, t)\, dt < +\infty$ for a.e. $s \in G$, satisfying the inequality

$$\varphi(t - s, u) \leqslant \varphi(t, cu) + \xi(s, t)$$

for s, $t \in G$ and $u \geqslant 0$. If, moreover, there holds $\int_G \xi(s, t)\, dt \to 0$ as $s \to 0$, then φ will be called *τ-bounded.*

Now, let $\mathscr{A} \in \mathscr{L}(\mathbb{R}^n, \mathbb{R})$ be fixed; we have the following:

THEOREM 2. *Let φ be a weakly τ-bounded N-function depending on a parameter $t \in G$. Let K be an $(L, \psi)_0$-Lipschitz kernel function with respect to $\mathscr{A}$, where $L(\cdot)\, \psi(\cdot, 1) \in L^1(C) \cap L^{\varphi^*}(C)$, for every compact $C \subset G$, where φ^* is the N-function, complementary to φ in the sense of Young. Let*

$$(Tf)(s) = \int_G K(t, f(s+t), f'(s+t), ..., f^{(n-1)}(s+t))\, dt \tag{3}$$

for $s \in G$ and let Dom T be the domain of the operator T. Then

$$W^{\varphi}_{n-1}(G) \subset \text{Dom } T.$$

Proof. We will consider the case when $G = \mathbb{R}$ and the function f has compact support. First, we prove that

$$\int_G |K(t, D_n f(s+t))|\, dt < +\infty \tag{4}$$

for a.e. $s \in G$, supposing $f \in W^{\varphi}_{n-1}(G)$. Let $f \in \mathscr{L}^0(G)$ be a fixed function. Since f has compact support, also the function $f(s + \cdot)$ has compact support for a.e. $s \in \mathbb{R}$; we denote by R_s an interval containing such support. For such $s \in \mathbb{R}$ let us write

$$A = \{t \in G : \|D_n f(t+s)\|_{\mathbb{R}^n} > 1\}, \qquad A' = G \setminus A.$$

Since K is a kernel function with respect to $\mathscr{A}$, we obtain

$$\begin{aligned}&\int_G |K(t, D_n f(s+t))|\, dt\\ &\qquad = \int_{R_s} |K(t, D_n f(s+t))|\, dt\\ &\qquad \leqslant \int_A L(t)\, \psi(t, \|D_n f(s+t)\|_{\mathbb{R}^n})\, dt + \int_{R_s} L(t)\, \psi(t, 1)\, dt.\end{aligned}$$

We have only to prove that the first term at the right-hand side of the above inequality is finite. By concavity of ψ we have $\psi(t,u)\leqslant\psi(t,1)\,u$, for $u>1$. Hence

$$\int_A L(t)\,\psi(t,\|D_nf(s+t)\|_{\mathbb{R}^n})\,dt\leqslant\int_A L(t)\,\psi(t,1)\,\|D_nf(s+t)\|_{\mathbb{R}^n}\,dt$$
$$\leqslant\int_A L(t)\,\psi(t,1)\sum_{k=0}^{n-2}|f^{(k)}(s+t)|\,dt$$
$$+\int_A L(t)\,\psi(t,1)\,|f^{(n-1)}(s+t)|\,dt.$$

Now $\sum_{k=0}^{n-2}|f^{(k)}(s+t)|$ is continuous and bounded as a function of the variable $t\in\mathbb{R}$. Consequently the first term at the right-hand side of the above inequality is finite. Taking in the Young inequality, $uv\leqslant\varphi^*(t,u)+\varphi(t,v)$,

$$u=\lambda L(t)\,\psi(t,1),\qquad v=\lambda\,|f^{(n-1)}(t)|,$$

for $t\in G=\mathbb{R}$ and $\lambda>0$, we obtain

$$\int_A L(t)\,\psi(t,1)\,|f^{(n-1)}(s+t)|\,dt\leqslant\frac{1}{\lambda^2}\int_{R_s}\varphi^*(t,\lambda L(t)\,\psi(t,1))\,dt$$
$$+\frac{1}{\lambda^2}\int_{R_s}\varphi(t,\lambda\,|f^{(n-1)}(s+t)|)\,dt.$$

The first term at the right-hand side of the last inequality is finite for sufficiently small $\lambda>0$, because $L(\cdot)\,\psi(\cdot,1)\in L^{\varphi^*}(C)$, for every compact $C\subset G$. In order to estimate the second term, we apply the weak τ-boundedness of φ and we obtain

$$\int_{R_s}\varphi(t,\lambda\,|f^{(n-1)}(s+t)|)\,dt$$
$$=\int_{R_s+s}\varphi(t-s,\lambda\,|f^{(n-1)}(t)|)\,dt$$
$$\leqslant\int_a^b\varphi(t,\lambda c\,|f^{(n-1)}(t)|)\,dt+\int_a^b\xi(s,t)\,dt<+\infty$$

for sufficiently small $\lambda>0$ and where $[a,b]$ is any compact interval which contains R_s+s, for the fixed $s\in\mathbb{R}$. Consequently we obtain the relation (4). Applying the inequality (4) to the positive and negative parts of K and applying the Fubini–Tonelli theorem, we obtain $f\in\operatorname{Dom}T$.

Remark 1. When the N-function φ satisfies the condition $\varphi(t, u)\, u^{-1} \to +\infty$ as $u \to +\infty$ uniformly with respect to t on each compact set C, we have $L^{\varphi^*}(C) \subset L^1(C)$, for any compact $C \subset \mathbb{R}$, and we may replace the assumption $L(\cdot)\, \psi(\cdot, 1) \in L^1(C) \cap L^{\varphi^*}(C)$ by $L(\cdot)\, \psi(\cdot, 1) \in L^{\varphi^*}(C)$. For example, this always happens when φ does not depend on the parameter t.

5. AN EMBEDDING THEOREM FOR THE OPERATOR T

We are going now to give an embedding theorem for the operator T. As before G denotes the real line $\mathbb{R}$ (by considering functions with compact support) or $G = [a, b]$ (by considering periodic extensions of the involved functions).

THEOREM 3. *Let the assumption of Theorem* 2 *be satisfied and let* ρ *be a modular in* $L^0(G)$, *quasimonotone with a constant* $M > 0$, $\mathcal{J}$-*convex, and such that the triple* $\{\rho, \psi, \bar{\eta}\}$ *is properly directed, where* $\bar{\eta}$ *is the extension of the modular* η *defined by* (2), *as described in the previous section. Then the operator* T *defined by* (3) *maps the space* $W^{\varphi}_{n-1}(G)$ *into the modular space* $L^0_{\rho}(G)$ *and*

$$\rho(aTf) \leqslant M[\eta(\lambda cf) + h_0] \tag{5}$$

for $f \in W^{\varphi}_{n-1}(G)$, $0 < \lambda < 1$, *and* $0 < a < C_{\lambda}(DM)^{-1}$.

Proof. Since φ is weakly τ-bounded, so the modular η defined by (2) is τ-subbounded, because for $f \in W^{\varphi}_{n-1}(G)$ we have

$$\begin{aligned}
\eta(f(\cdot + t)) &= \sum_{k=0}^{n-1} \int_G \varphi(s, |f^{(k)}(s+t)|)\, ds \\
&\leqslant \sum_{k=0}^{n-1} \int_G \varphi(s, c\, |f^{(k)}(s)|)\, ds + n \int_G \xi(t, s)\, ds \\
&\leqslant \eta(cf) + h_0,
\end{aligned}$$

where $h_0 = n \sup_{t \in G} \int_G \xi(t, s)\, ds$. Consequently we have, for every $0 < \lambda < 1$ and $a > 0$,

$$\begin{aligned}
\rho(aTf) &\leqslant M\rho \left[\int_G p(t)\, aDM\psi(t, \|D_n f(\cdot + t)\|_{\mathbb{R}^n})\, dt \right] \\
&\leqslant M \int_G p(t)\, \rho(aDM\psi(t, \|D_n f(\cdot + t)\|_{\mathbb{R}^n}))\, dt
\end{aligned}$$

$$\leqslant M \int_G p(t)\, \bar{\eta}(\lambda D_n f(\cdot + t))\, dt$$

$$\leqslant M\eta(\lambda c f) + M h_0 < +\infty,$$

that is (5); as a direct consequence, $Tf \in L^0_\rho(G)$.

6. APPROXIMATION THEOREMS IN $W^{\varphi}_{n-1}(G)$

We are going now to formulate the approximation theorem (Theorem 1) in the special case of generalized Orlicz–Sobolev spaces $W^{\varphi}_{n-1}(G)$, taking the modular η defined by formula (2) and defining the linear continuous functional $\mathscr{A}: \mathbb{R}^n \to \mathbb{R}$ by $\mathscr{A}(u_0, ..., u_{n-1}) = u_0$. If φ is weakly τ-bounded with a function ξ, then η is τ-subbounded with a constant $h_0 = n \sup_{t \in G} \int_G \xi(t, s)\, ds$ (see the proof of Theorem 3).

Let us also remark that $K(t, u_0, u_1, ..., u_{n-1}) = 0$ if $u_0 = \mathscr{A}u = 0$; that is, $K(t, 0, u_1, ..., u_{n-1}) = 0$, for arbitrary $t \in G$ and $u_1, ..., u_{n-1} \in \mathbb{R}$. Thus Theorem 1 yields the following

THEOREM 4. *Let G be defined as in Section 5. Let ρ be a $\mathscr{J}$-convex, quasimonotone (with constant $M \geqslant 1$) modular on $L^0(G)$. Let η be the modular defined by (2), generated by a weakly τ-bounded, φ-function φ, and let $\bar{\eta}$ be the extension of the modular η as defined in Section 4. Let $\{\rho, \psi, \bar{\eta}\}$ be a properly directed triple, and let $K: G \times \mathbb{R}^n \to \mathbb{R}$ be an (L, ψ)-Lipschitz kernel function, where $K(t, 0, u_1, ..., u_{n-1}) = 0$, for any $t \in G$ and $u_1, ..., u_{n-1} \in \mathbb{R}$. Then for any $f \in \mathrm{Dom}\, T \cap W^{\varphi}_{n-1}$, $U \in \mathscr{U}$, $\lambda \in]0, 1[$, and $a \in]0, C_\lambda (2DM)^{-1}[$ there holds the inequality*

$$\rho[a(Tf - f)] \leqslant M\omega_\eta(\lambda f, U) + M[2\eta(2\lambda c f) + h_0] \times \int_{G \setminus U} p(t)\, dt + M\rho(2aMr_0 f),$$

where $D = \int_G L(t)\, dt$, $p(t) = L(t)\, D^{-1}$,

$$r_0 = \sup_{u_0 \neq 0} \left| \frac{1}{u_0} \int_G K(t, u_0, ..., u_{n-1})\, dt - 1 \right|,$$

and h_0 is the constant of the τ-subboundedness of η.

Now we are going to investigate the η-modulus of continuity ω_η for the modular defined in (2). As $\mathscr{U}$ we take the family of intervals $U_\delta = [-\delta, \delta]$, where $\delta > 0$. Then we have for an arbitrary $\lambda > 0$ and $f \in W^{\varphi}_{n-1}(G)$,

$$\omega_\eta(\lambda f, U_\delta) = \sup_{|t| \leqslant \delta} \left\{ \sum_{k=0}^{n-2} \int_G \varphi(s, \lambda\, |f^{(k)}(t+s) - f^{(k)}(s)|)\, ds \right.$$
$$\left. + \int_G \varphi(s, \lambda\, |f^{(n-1)}(t+s) - f^{(n-1)}(s)|)\, ds \right\}$$
$$\leqslant \sum_{k=0}^{n-2} \int_G \varphi(s, \lambda\omega(f^{(k)}, \delta))\, ds + \omega_{\eta_1}(\lambda f^{(n-1)}, \delta), \qquad (6)$$

where $\omega(g, \delta) = \sup_{|t| \leqslant \delta} \sup_{s \in G} |g(t+s) - g(s)|$ and

$$\eta_1(g) = \int_G \varphi(t, |g(t)|)\, dt.$$

We recall that a φ-function φ is called *locally integrable* on G if $\varphi(\cdot, u) \in L^1(C)$ for every compact set $C \subset G$, for any $u \in \mathbb{R}_0^+$. Under the notations of Section 5 there holds the following:

PROPOSITION 2. *Let φ be a τ-bounded, locally integrable, φ-function depending on a parameter. Then for every function $f \in W_{n-1}^{\varphi}(G)$ there exists a number $\lambda > 0$ such that*:

$$\lim_{\delta \to 0} \omega_\eta(\lambda f, U_\delta) = 0.$$

Proof. Suppose that $G = \mathbb{R}$ has f has a compact support C. Since $f^{(k)} \in C(G)$, for $k = 0, 1, ..., n-2$, so $\omega(f^{(k)}, \delta) \to 0$ as $\delta \to 0$ for $k = 1, ..., n-2$. Let us take $\delta_0 > 0$ so small that $\omega(f^{(k)}, \delta_0) < 1$. Let $D \subset G$ be a compact set such that $[-\delta_0, \delta_0] + C \subset D$. So $f^{(k)}(s+t) - f^{(k)}(s) = 0$ for $s \notin D$, for every $t \in [-\delta_0, \delta_0]$. Thus for the above indices k, we have for $0 < \delta < \delta_0$

$$\sum_{k=0}^{n-2} \int_G \varphi(s, \lambda\omega(f^{(k)}, \delta))\, ds \leqslant \sum_{k=0}^{n-2} \omega(f^{(k)}, \delta) \int_D \varphi(s, \lambda)\, ds \to 0$$

as $\delta \to 0$, by the local integrability of φ, for every $\lambda > 0$. Applying the inequality (6) it is enough to prove that $\omega_{\eta_1}(\lambda f, \delta) \to 0$ as $\delta \to 0$ for $g \in L^\varphi(G)$ and sufficiently small $\lambda > 0$. Here $L^\varphi(G)$ is the Musielak–Orlicz space generated by the function φ. Now, since $\int_D \varphi(s, u)\, ds < +\infty$ for all $u \geqslant 0$, the modular η_1 is monotone, absolutely finite, and absolutely continuous modular on $L^\varphi(G)$ such that $\eta_1(f(t + \cdot)) \leqslant \eta_1(cf) + h(t)$ with some $c \geqslant 1$, and $h(t) \to 0$ as $t \to 0$, and $f \in L^0(G)$ (see [3]). By [3, Theorem 2], we obtain $\omega_{\eta_1}(\lambda g, \delta) \to 0$ as $\delta \to 0$ for sufficiently small $\lambda > 0$.

Remark 2. We remark that the previous theorem can be proved also when $G=[a,b]$ and the involved functions are extended by periodicity outside the interval $[a,b]$. Moreover, we point out that if in Theorem 4 the assumptions of Theorem 2 are satisfied, then one may replace the requirement $f \in \operatorname{Dom} T \cap W^{\varphi}_{n-1}$ with simply $f \in W^{\varphi}_{n-1}$.

Theorem 4 and Proposition 2 may be applied in order to obtain a convergence theorem. Arguing as, for example, in [2], we introduce an abstract nonempty set W of indices w filtered by a family $\mathscr{W}$ of its subsets. In place of one kernel function K we take a family $\mathbb{K}=(K_w)_{w \in W}$ of kernel functions, which we call a *kernel*. We assume that $K_w(t, u_0, u_1, ..., u_{n-1})=0$ whenever $u_0=0$, for all $w \in W$. Let $\mathbb{L}=(L_w)_{w \in W}$ be a family of nonnegative functions $L_w \in L^1(G)$ and let $D_w=\int_G p_w(t)\,dt$, $p_w(t)=L_w(t)\,D_w^{-1}$. The kernel $\mathbb{K}$ is said to be $(\mathbb{L}, \psi)$-Lipschitz if the kernel functions K_w are (L_w, ψ)-Lipschitz, for all $w \in W$. Let $D=\sup_{w \in W} D_w < +\infty$. We say that the kernel $\mathbb{K}$ is *strongly singular* if

$$\int_{G \backslash U_\delta} p_w(t)\,dt \xrightarrow{\mathscr{W}} 0$$

and

$$r_0(w)=\sup_{u_0 \neq 0} \left| \frac{1}{u} \int_G K_w(t, u_0, ..., u_{n-1})\,dt - 1 \right| \xrightarrow{\mathscr{W}} 0,$$

where $\xrightarrow{\mathscr{W}}$ represents the convergence with respect to the filter $\mathscr{W}$. Denote now by $\mathbf{T}=(T_w)$ the corresponding family of operators

$$(T_w f)(s)=\int_G K_w(t, D_n f(s+t))\,dt$$

for $w \in \mathscr{W}$ and $f \in \operatorname{Dom} \mathbf{T}=\bigcap_{w \in W} \operatorname{Dom} T_w$. Then the following theorem may be derived, applying Theorem 4 and Proposition 2, in an analogous manner as in [2, Theorem 2]:

THEOREM 5. *Let G be as in Section* 5. *Let ρ be a $\mathscr{J}$-convex, quasimonotone modular on $L^0(G)$. Let φ be a τ-bounded, locally integrable φ-function depending on a parameter, let η be the modular defined on $W^{\varphi}_{n-1}(G)$ by the formula* (2), *and let us suppose that the triple $\{\rho, \psi, \bar{\eta}\}$ is properly directed. Let $\mathbb{K}=(K_w)_{w \in W}$ be a strongly singular, $(\mathbb{L}, \psi)$-Lipschitz kernel such that $K(t, 0, u_1, ..., u_{n-1})=0$ for $t \in G$ and $u_1, ..., u_{n-1} \in \mathbb{R}$. Let $f \in L^0_\rho(G) \cap W^{\varphi}_{n-1}(G) \cap$* Dom T. *Then*

$$\rho[a(T_w f - f)] \xrightarrow{\mathscr{W}} 0$$

for sufficiently small $a>0$.

Remarks. 1. Let us remark that if additionally there are satisfied the assumptions of Theorem 2 then one may replace the requirement $f \in L^0_\rho(G) \cap W^\varphi_{n-1}(G) \cap \text{Dom}\, T$ simply by $f \in L^0_\rho(G) \cap W^\varphi_{n-1}(G)$.

2. Theorem 5 remains true if we replace the strong singularity of the kernel by the *singularity*, i.e.,

$$r_k(w) = \sup_{1/k \leqslant |u_0| \leqslant k} \left| \frac{1}{u_0} \int_G K_w(t, u_0, ..., u_{n-1})\, dt - 1 \right| \xrightarrow{\mathscr{W}} 0$$

for $k = 1, 2, ...$ In this case we have to assume that the modular ρ is finite and absolutely continuous (see also [2]).

3. The theory developed in this paper can be easily generalized by relaxing the $\mathscr{J}$-convexity of $\tilde{\rho}$. Indeed we can assume that $\tilde{\rho}$ is M-quasi-convex. This concept was introduced in [4]; we remark that the related concept for functions was given in [6, 7].

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